# CLASSICAL PROOFS OF KATO TYPE SMOOTHING ESTIMATES FOR THE SCHRÖDINGER EQUATION WITH QUADRATIC POTENTIAL IN $\mathbb{R}^{n+1}$ WITH APPLICATION

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Abstract. In this paper, we consider the Schrödinger equation with quadratic potential

$$i\frac{\partial}{\partial t}u=-\triangle u+|x|^2\,u\ in\ \mathbb{R}^{n+1},\ u(x,0)=f(x)\in L^2(\mathbb{R}^n).$$

Using Hermite functions and some other classical tools, we give an elementary proof of the Kato type smoothing estimate: for $i\neq j\neq k$, $\delta\in[0,1]$, and $n\geqslant 3$

$$\int_0^{2\pi}\int_{\mathbb{R}^n}\frac{|u(x,t)|^2}{\left(x_i^2+x_j^2+x_k^2\right)^{\delta}}dxdt\leqslant C\,\|f\|_2^2\,.$$

This is equivalent to proving a uniform $L^2(\mathbb{R}^n)$ boundedness result for a family of singularized Hermite projection kernels.

As an application of the above estimate, we also prove the $\mathbb{R}^9$ collapsing variable type Strichartz estimate

$$\int_0^{2\pi}\int_{\mathbb{R}^3}|u(\mathbf{x},\mathbf{x},\mathbf{x},t)|^2\,d\mathbf{x}dt\leqslant C\left\|(-\triangle+|x|^2)f\right\|_2^2$$

where $\mathbf{x}\in\mathbb{R}^3$.

## 1. Introduction

In Bose-Einstein condensation (BEC), particles of integer spins ("Bosons") occupy a macroscopic quantum state often called the "condensation". In early lab experiments of BEC [1] [10], the particles were kept together by use of trapping potentials created by the effect of a magnetic field on the particle spins. In principle, the magnetic field has a complicated spatial structure. The interaction of the magnetic field with the spin is conveniently modeled by a quadratic potential. This captures salient features of the actual trap, especially the property that the external potential rises at large distances. In later experiments, e.g., [23], the trapping potential is produced by complicated laser fields; but mathematically, one can still use a quadratic potential as a simplified yet generic model. So the spin of the particle is removed in modeling and the effect of a trap is included in the form of a quadratic external potential. This physical background suggests that we study the Schrödinger equation with quadratic potential

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$$i\frac{\partial}{\partial t}u = -\triangle u + |x|^2 u \text{ in } \mathbb{R}^{n+1}, \tag{1.1}$$

with initial data

$$u(x,0) = f(x) \in L^2(\mathbb{R}^n).$$

Many aspects of equation 1.1 which came from the study of the free Schrödinger equation

$$i\frac{\partial}{\partial t}\phi = -\triangle\phi \text{ in } \mathbb{R}^{n+1} \tag{1.2}$$

have been studied by several authors. Its Strichartz estimates were proved by Koch and Tataru [4], Carles [6], Nandakumarana and Ratnakuma [19]. The well-posedness of its nonlinear energy critical version with radial initial data was studied by Killip, Visan and Zhang [15]. Bongioanni and Torrea, and Bongioanni and Rogers, proved results on the pointwise convergence to the initial data in [3] and [4]. Concerning the Kato $\frac{1}{2}$-smoothing effect, Doi (and later Bongioanni and Rogers) proved

$$\int_0^T\int_{\mathbb{R}^n} \frac{\left|(I-(-\triangle+|x|^2))^{\frac{1}{4}}u(x,t)\right|^2}{(1+|x|^2)^{\frac{1}{2}+\varepsilon}}dxdt \leqslant C\|f\|_2^2 \tag{1.3}$$

in [9] ([4]), Robbiano and Zuily proved

$$\int_0^T\int_{\Omega} \left|\chi(x)(I-(-\triangle+|x|^2))^{\frac{1}{4}}u(x,t)\right|^2 dxdt \leqslant C\|f\|_2^2$$

for an external domain $\Omega$ and $\chi \in C_0^\infty(\overline{\Omega})$ in [21]. However, both [9] and [21] made extensive use of pseudo-differential techniques which did not suffice to prove the equation 1.1 counterpart to the Kato estimate

$$\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} \frac{|\phi(x,t)|^2}{|x|^2}dxdt \leqslant C\|\phi(\cdot,0)\|_2^2 \tag{1.4}$$

in Kato and Yajima [14], or its generalization

$$\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} \frac{\left||\nabla|^\alpha\phi(x,t)\right|^2}{|x|^{2-2\alpha}}dxdt \leqslant C\|\phi(\cdot,0)\|_2^2, \text{ for } \alpha\in[0,\frac{1}{2}) \tag{1.5}$$

in Kato and Yajima [14], and Ben-Artzi and Klainerman [2], where $\phi$ is the solution to the free Schrödinger equation 1.2 in the case $n \geqslant 3$.

**Remark 1.** *Doi proved 1.3 type estimates in the case involving variable coefficients. Bongioanni and Rogers proved an estimate similar to 1.3 for equation 1.1 using Hermite functions. Their paper contains a series of results parallel to those in Vega* [26].

**Remark 2.** *The expository note* [7] *gives extensions of estimate 1.5 to a class of dispersive equations, and simultaneously arrives at the optimal constant for each $\alpha$ and $n$. The fact that $\frac{\pi}{n-2}$ is the best constant achievable for the $\alpha = 0$ case is due to Simon* [22].

This paper aims to prove Kato type smoothing estimates similar to 1.4 when $n \geqslant 3$ for equation 1.1 without using any pseudo-differential techniques.

In fact, we have

**Theorem 1.** *Let $u$ be the solution to equation 1.1 in the case $n \geqslant 3$, then for $\delta \in [0,1]$ and $i \neq j \neq k$, one has the estimate*

$$\int_0^{2\pi} \int_{\mathbb{R}^n} \frac{|u(x,t)|^2}{\left(x_i^2 + x_j^2 + x_k^2\right)^{\delta}} dx dt \leqslant C \|f\|_2^2 . \tag{1.6}$$

*In particular, when $\delta = 1$, the above estimate implies the Kato estimate 1.4 for equation 1.1 in the case $n \geqslant 3$ because of the trivial inequality $\left(x_i^2 + x_j^2 + x_k^2\right) \leqslant |x|^2$.*

**Remark 3.** *$u$ naturally has period $2\pi$ in the time variable $t$. We will show this in section 2. This was also shown in Nandakumarana and Ratnakuma* [19].

**Remark 4.** *Without lose of generality, from here on out we assume $i = 1$, $j = 2$, $k = 3$ for simplicity since the general case has an identical proof.*

**Remark 5.** *We shall also point out that the Kato type estimate*

$$\int_0^{2\pi} \int_{\mathbb{R}^n} \frac{\left||\nabla|^{\alpha} u(x,t)\right|^2}{|x|^{2-2\alpha}} dx dt \leqslant C \|f\|_2^2 , \text{ for } \alpha \in [0, \frac{1}{2})$$

*which is the equation 1.1 counterpart to estimate 1.5 is still unproven.*

As an application of theorem 1, we have the following collapsing variable type Strichartz estimate.

**Theorem 2.** *Let $u(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t)$ solves equation 1.1 in the case $n = 9$ with $\mathbf{x}_i \in \mathbb{R}^3$, then one has the estimate*

$$\int_0^{2\pi} \int_{\mathbb{R}^3} |u(\mathbf{x}, \mathbf{x}, \mathbf{x}, t)|^2 d\mathbf{x} dt \leqslant C \left\|(-\triangle + |x|^2) f\right\|_2^2 . \tag{1.7}$$

**Remark 6.** *The ordinary Strichartz estimate gives*

$$\|\phi\|_{L_t^6 L_x^6} \leqslant C \|\phi(\cdot, 0)\|_{\dot{H}^{\frac{2}{3}}}$$

*for $\phi$ satisfying equation 1.2 in the case $n = 3$. This leads us to consider estimate 1.7. Estimates similar to 1.7 were also considered in Grillakis and Margetis* [13], *and Klainerman and Machedon* [17] *in the setting of interacting Boson systems.*

**Remark 7.** *In Bongioanni and Torrea* [3], *Bongioanni and Rogers* [4], *and Thangavelu* [25], *$\left\|(-\triangle + |x|^2)^{\frac{s}{2}} f\right\|_2^2$ is called the Hermite-Soblev $\mathcal{H}^s$ norm of $f$. In sections 4 and 5, we will need the following lemma proved by Thangavelu concerning Hermite-Soblev spaces in* [25].

**Lemma 1.** [25] *The operator $(I - \triangle)^{\frac{s}{2}} (-\triangle + |x|^2)^{-\frac{s}{2}}$ is bounded on $L^2(\mathbb{R}^n)$ for $s \geqslant 0$, or in other words*

$$\left\|(I - \triangle)^{\frac{s}{2}} f\right\|_2 \leqslant C \left\|(-\triangle + |x|^2)^{\frac{s}{2}} f\right\|_2 , \ s \geqslant 0.$$

Theorem 1 will be deduced from the theorems below.

**Theorem 3.** *Let $u$ be the solution to equation 1.1 in the case $n \geqslant 2$, then $\forall \delta \in [0,1)$, one has the estimate*

$$\int_0^{2\pi} \int_{\mathbb{R}^n} \frac{|u(x,t)|^2}{(x_1^2 + x_2^2)^\delta} dxdt \leqslant C \|f\|_2^2 ,$$

*which implies estimate 1.6 in the case $n \geqslant 3$ and $\delta \in [0,1)$.*

**Theorem 4.** *Say $g(-x) = -\, g(x) \in L^2(\mathbb{R})$, then we have the equality*

$$\int_0^{2\pi} \int_{\mathbb{R}} \frac{\left| e^{-it(-\triangle + |x|^2)} g \right|^2}{|x|^2} dxdt = 4\pi \|g\|_2^2 . \tag{1.8}$$

*In particular, estimate 1.8 is equivalent to estimate 1.6 in the 3d radial case i.e.*

$$\int_0^{2\pi} \int_{\mathbb{R}^3} \frac{\left| e^{-it(-\triangle + |x|^2)} \psi \right|^2}{|x|^2} dxdt = 4\pi \|\psi\|_{L^2(\mathbb{R}^3)}^2$$

*if $\psi$ is a $L^2(\mathbb{R}^3)$ radial function.*

**Remark 8.** *There is an identity similar to equality 1.8 for the free Schrödinger equation 1.2. See the expository note [7].*

**Theorem 5.** *Say $d(\pm x_1, \pm x_2, \pm x_3) = d(x_1, x_2, x_3) \in L^2(\mathbb{R}^3)$, then one has the estimate*

$$\int_0^{2\pi} \int_{\mathbb{R}^3} \frac{\left| e^{-it(-\triangle + |x|^2)} d \right|^2}{|x|^2} dxdt \leqslant C \|d\|_2^2 .$$

Because we can write $f$ as a sum of its $x_1$-odd part and $x_1$-even part by defining

$$f_{odd}(x) = \frac{f(x_1, x_2, x_3) - f(-x_1, x_2, x_3)}{2}$$

and

$$f_{even}(x) = \frac{f(-x_1, x_2, x_3) + f(x_1, x_2, x_3)}{2}.$$

So

$$f(x) = f_{odd}(x) + f_{even,odd}(x) + f_{even,even,odd}(x) + f_{even,even,even}(x) \tag{1.9}$$

if we iterate the procedure three times. The linearity of equation 1.1 and the fact that the terms in 1.9 are all linear combinations of $f$ shows that estimate 1.6 in the case when $n = 3$ and $\delta = 1$ indeed follows from theorems 4 and 5.

Moreover, theorems 1 and 3 are equivalent to the following uniform $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ boundedness result for a family of singularized Hermite projection kernels.

**Theorem 6.** *For $n \geqslant 3$ and $\delta \in [0,1]$, ( $\delta \in [0,1)$ when $n = 2$ ), the singularized Hermite projection kernels $\left\{ \frac{\Phi_k(x,y)}{|x|^\delta} \right\}_k$ map $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ uniformly where $\Phi_k$ is the usual Hermite projection kernel with respect to the $k$-eigenspace defined in lemma 4, in other words, there exists a $C > 0$ depending only on $\delta$ and $n$ such that*

$$\left\| \int_{\mathbb{R}^n} \frac{\Phi_k(\cdot, y)}{|\cdot|^\delta} f(y) dy \right\|_2 \leqslant C \|f\|_2 .$$

*Moreover the more singular family* $\left\{\frac{\Phi_k(x,y)}{|x|^\delta|y|^\delta}\right\}_k$ *also maps* $L^2(\mathbb{R}^n)$ *to* $L^2(\mathbb{R}^n)$ *uniformly via the standard* $TT^*$ *method.*

Regularized Hermite projection kernels were studied in [3], [19], [20], and [24]. But, to the best of the author's knowledge, theorem 6 might be the first result on the singularized Hermite projection kernels.

## 2. Some basics of Hermite functions and the proof of theorem 3

To prove theorems 1, 3, 4, 5 and 6, we will need the Hermite functions and some of their properties. For more details outside of lemma 5 whose proof is provided in the appendix I, we refer the reader to Thangavelu's monograph [24].

**Definition 1.** [24] *We define an* $n$ *dimensional Hermite function* $\Phi_\alpha(x)$ *where* $\alpha$ *is an* $n$*-multiindex by*

$$\Phi_\alpha(x)=\prod_{i=1}^n h_{\alpha_i}(x_i),$$

*where* $h_k$ *are the one dimensional normalized Hermite functions defined by*

$$h_k(t)=\frac{(-1)^k}{(2^k k!\sqrt{\pi})^{\frac12}}e^{\frac{t^2}{2}}\frac{d^k}{dt^k}(e^{-t^2}),\ t\in\mathbb{R}.$$

Then we have the following well-known properties.

**Lemma 2.** [24] $\Phi_\alpha$ *are the eigenfunctions of the Fourier transform with eigenvalues* $(-i)^{|\alpha|}$ *i.e.*

$$\widehat{\Phi_\alpha}(\xi)=(-i)^{|\alpha|}\Phi_\alpha(\xi)$$

**Lemma 3.** [24] $\Phi_\alpha$ *are also the eigenfunctions of the Hermite operator* $-\triangle+|x|^2$ *with eigenvalues* $2|\alpha|+n$*. Moreover they form an orthonormal basis of* $L^2(\mathbb{R}^n)$*.*

As this lemma states, we can write

$$u(x,t)=\sum_\alpha e^{-i\lambda_\alpha t}a_\alpha\Phi_\alpha(x), \tag{2.1}$$

where $a_\alpha$ are the Fourier-Hermite coefficients

$$a_\alpha=\int_{\mathbb{R}^n}f(x)\Phi_\alpha(x)dx,$$

with convergence in $L^2(\mathbb{R}^n)$, i.e. $u$ is naturally periodic $2\pi$ in the time variable $t$ and we have

$$\begin{aligned}\int_0^{2\pi}|u(x,t)|^2\,dt &= \int_0^{2\pi}\sum_{\alpha,\beta}e^{-i(\lambda_\alpha-\lambda_\beta)t}a_\alpha\overline{a_\beta}\Phi_\alpha(x)\Phi_\beta(x)dt\\ &= 2\pi\sum_{\substack{\alpha,\beta\\ \lambda_\alpha=\lambda_\beta}}a_\alpha\overline{a_\beta}\Phi_\alpha(x)\Phi_\beta(x).\end{aligned}$$

But

$$\int_{\mathbb{R}}\sum_{\substack{\alpha,\beta\\ \lambda_\alpha=\lambda_\beta}}a_\alpha\overline{a_\beta}\Phi_\alpha(x)\Phi_\beta(x)dx_j=\sum_{\substack{\alpha,\beta\\ \lambda_\alpha=\lambda_\beta}}\delta_{a_j\beta_j}a_\alpha\overline{a_\beta}\prod_{\substack{i=1\\ i\neq j}}^n h_{\alpha_i}(x_i)\prod_{\substack{i=1\\ i\neq j}}^n h_{\beta_i}(x_i).$$

that is:

$$\int_0^{2\pi}\int_{\mathbb{R}^{n+1}} \frac{|u(x,t)|^2}{|(x,x_{n+1})|^{2\delta}} dx dx_{n+1} dt \leqslant 2\pi \int_{\mathbb{R}^n} \frac{1}{|x|^{2\delta}} \sum_{\substack{\alpha,\beta \\ \lambda_\alpha=\lambda_\beta}} \delta_{a_j\beta_j} a_\alpha \overline{a_\beta} \prod_{i=1}^n h_{\alpha_i}(x_i) \prod_{i=1}^n h_{\beta_i}(x_i) dx$$

hence the estimate

$$\int_0^{2\pi}\int_{\mathbb{R}^2} \frac{|u(x,t)|^2}{|x|^{2\delta}} dx dt \leqslant C \|f\|_2^2 \tag{2.2}$$

for $\delta \in [0,1)$ implies theorem 3 and we only need to prove theorem 1 in the case $n=3$ and $\delta=1$. We can now prove estimate 2.2 by the following lemma.

**Lemma 4.** [24] *Let $P_k$ be the Hermite projector corresponding to the $k$-eigenspace with kernel*

$$\Phi_k(x,y) = \sum_{|\alpha|=k} \Phi_\alpha(x)\Phi_\alpha(y)$$

*then there is a constant $C \geqslant 0$ independent of $k$ and $x$ such that*

$$|\Phi_k(x,x)| \leqslant C k^{\frac{n}{2}-1}. \tag{2.3}$$

Therefore

$$\begin{aligned}
& \int_0^{2\pi}\int_{\mathbb{R}^2} \frac{|u(x,t)|^2}{|x|^{2\delta}} dx dt & (2.4) \\
= \;& 2\pi \sum_k \int_{\mathbb{R}^2} \frac{|P_k f|^2}{|x|^{2\delta}} dx \\
\leqslant \;& 2\pi \sum_k \int_{\mathbb{R}^2-\mathbb{D}} |P_k f|^2 dx + 2\pi \sum_k \int_{\mathbb{D}} \frac{|P_k f|^2}{|x|^{2\delta}} dx \\
\leqslant \;& 2\pi \|f\|_2^2 + 2\pi \sum_k \int_{\mathbb{D}} \frac{1}{|x|^{2\delta}} \Bigg( \sum_{\substack{\alpha,\beta \\ |\alpha|=|\beta|=k}} |a_\alpha|^2 |\overline{a_\beta}|^2 \Bigg)^{\frac{1}{2}} \Bigg( \sum_{\substack{\alpha,\beta \\ |\alpha|=|\beta|=k}} |\Phi_\alpha(x)\Phi_\beta(x)|^2 \Bigg)^{\frac{1}{2}} dx \\
\leqslant \;& 2\pi \|f\|_2^2 + C \sum_k \left( \|P_k f\|_2^2 \int_0^{2\pi}\int_0^1 \frac{1}{r^{2\delta-1}} dr d\theta \right) \\
\leqslant \;& C \|f\|_2^2 .
\end{aligned}$$

**Remark 9.** *In the above computation, we have also proved the Morawetz inequality*

$$\sup_{x\in\mathbb{R}^2} \int_0^{2\pi} |u(x,t)|^2 dt \leqslant C \|f\|_2^2$$

*which is identical to the well-known version for the free Schrödinger equation 1.2 in* [8]*. This is another Kato type smoothing estimate.*

**Remark 10.** *Estimate 2.3 is also the key ingredient to prove the regularized Hermite projection kernel estimates in* [20]*. But it does not yield theorem 1 in the case when $n=3$ and $\delta=1$. Lemma 5 will introduce a new tool for that purpose.*

As
$$\int_0^{2\pi}\int_{\mathbb{R}^3} \frac{|u(x,t)|^2}{|x|^{2\delta}} dxdt = 2\pi \sum_k \int_{\mathbb{R}^3} \frac{|P_k f|^2}{|x|^{2\delta}} dx,$$
theorem 6 implies theorems 1 and 3. However, the fact that $e^{-i(2k+n)t} P_k f$ satisfies equation 1.1 with $u(x,0) = P_k f(x)$ shows that theorems 1 and 3 also imply theorem 6.

We are left with the proofs of theorems 4 and 5 which will need the following tool.

**Lemma 5.** *We define the "antiderivatives" of the 1-d Hermite functions to be*
$$X_{2k+1}(x) = \int_{-\infty}^{x} h_{2k+1}(t)dt$$
*and*
$$X_{2k}(x) = \int_{-\infty}^{x} sign(t) h_{2k}(t)dt$$
*which are by definition absolutely continuous. Moreover*
$$\int_{\mathbb{R}} (X_{2k+1}(x))^2 dx = 2 \tag{2.5}$$
*and*
$$\begin{aligned} \int_{\mathbb{R}} (X_{2k}(x))^2 dx &= 2(-1 + \sqrt{2} \sum_{i=0}^{k} \binom{\frac{1}{2}}{i}) \\ &\leqslant 3 \end{aligned} \tag{2.6}$$
*where*
$$\sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} = \sqrt{2}$$
*i.e.* $\lim_{k\to\infty} \|X_{2k}\|_2^2 = 2$ *and* $X_k \in H^1(\mathbb{R})$.

To the best of our knowledge, lemma 5 is new. The proof, which is a direct computation, is provided in the appendix I for completion. Now we can give the proofs of theorems 4 and 5.

## 3. Proof of theorem 4

We only need to prove
$$\int_{\mathbb{R}} \frac{|P_{2k+1} g|^2}{|x|^2} dx = 2\,|a_{2k+1}|^2$$
because $g(-x) = -\,g(x)$ implies $a_{2k} = 0$, $\forall k$. In fact since $h_{2k}(x)$ is even, we have
$$\begin{aligned} a_{2k} &= \int_{\mathbb{R}} g(x) h_{2k}(x) dx \\ &= 0. \end{aligned}$$

One notices that
$$h_{2k+1}(\xi) = \frac{d}{dx} X_{2k+1}(\xi)$$

i.e.

$$(-i)^{-(2k+1)}h_{2k+1}(x) = x\check{X}_{2k+1}(x)$$

hence

$$\begin{aligned}\int_{\mathbb{R}} \frac{|P_{2k+1}g|^2}{|x|^2}dx &= |a_{2k+1}|^2 \int_{\mathbb{R}} \left|\check{X}_{2k+1}(x)\right|^2 dx \\ &= |a_{2k+1}|^2 \int_{\mathbb{R}} |X_{2k+1}(x)|^2 dx \\ &= 2\,|a_{2k+1}|^2\end{aligned}$$

via equality 2.5. Whence we have deduced theorem 4.

## 4. Proof of theorem 5

It suffices to prove that there exists a $C > 0$ independent of $k$ s.t.

$$\int_{\mathbb{R}^3} \frac{|P_k d|^2}{|x|^2}dx \leqslant C\,\|P_k d\|_2^2\,.$$

Throughout this section, we will assume $k \neq 0$. In the case when $k = 0$, $P_k d$ has only one term

$$\begin{aligned}P_0 d &= a_0 h_0(x_1)h_0(x_2)h_0(x_3) \\ &= a_0\left(\sqrt{\pi}\right)^{-\frac{3}{2}} e^{-\frac{|x|^2}{2}}\end{aligned}$$

and hence is a 3d radial function, and we dealt with this situation in theorem 4. In fact, it is easy to compute that

$$\int_{\mathbb{R}^3} \frac{|P_0 d|^2}{|x|^2}dx = 2\,|a_0|^2$$

which matches theorem 4.

We write

$$\begin{aligned}P_k d(x) &= \sum_{\substack{\alpha \\ \alpha_1,\ \alpha_2,\ \alpha_3 \text{ are even} \\ |\alpha|=k}} a_\alpha h_{\alpha_1}(x_1)h_{\alpha_2}(x_2)h_{\alpha_3}(x_3) \\ &= \sum_{\alpha\in I} + \sum_{\alpha\in II} + \sum_{\alpha\in III}\end{aligned}$$

where

$$I = \{\alpha : |\alpha| = k,\ \alpha_1,\ \alpha_2,\ \alpha_3 \text{ are even, and } \alpha_1 \geqslant \frac{\alpha_2 +\ \alpha_3}{2}\}$$

$$II = \{\alpha : |\alpha| = k,\ \alpha_1,\ \alpha_2,\ \alpha_3 \text{ are even, and } \alpha_2 \geqslant \frac{\alpha_1 +\ \alpha_3}{2}\}$$

$$III = \{\alpha : |\alpha| = k,\ \alpha_1,\ \alpha_2,\ \alpha_3 \text{ are even, and } \alpha_3 \geqslant \frac{\alpha_1 +\ \alpha_2}{2}\}.$$

**Remark 11.** *Suppose we have $\alpha$ s.t. $\alpha_1 < \frac{\alpha_2+\ \alpha_3}{2}$, $\alpha_2 < \frac{\alpha_1+\ \alpha_3}{2}$, and $\alpha_3 < \frac{\alpha_2+\ \alpha_1}{2}$, then $\alpha_1+\alpha_2+\alpha_3 < \alpha_1+\alpha_2+\alpha_3$ which is a contradiction. So $I$, $II$, and $III$ covers all cases. In some cases, $I$, $II, III$ do not intersect trivially. In these cases, we just count the crossing terms once in one proper set. Moreover $a_\alpha = 0$, $\forall\alpha$ with one odd index due to $d(\pm x_1, \pm x_2, \pm x_3) = d(x_1, x_2, x_3)$, in fact since $h_{\alpha_1}(x_1)$ is odd if $\alpha_1$ is odd, we have*

$$\begin{aligned} a_\alpha &= \int_{\mathbb{R}^2} dx_2 dx_3 \int_{\mathbb{R}} dx_1 d(x) h_{\alpha_1}(x_1) h_{\alpha_2}(x_2) h_{\alpha_3}(x_3) \\ &= \int_{\mathbb{R}^2} dx_2 dx_3 h_{\alpha_2}(x_2) h_{\alpha_3}(x_3) \cdot 0 \\ &= 0 \end{aligned}$$

So it is enough to prove

$$\int\limits_{\mathbb{R}^3} \frac{\left|\sum_{\alpha\in I}\right|^2}{|x|^2} dx \leqslant C \sum_{\alpha\in I} |a_\alpha|^2, \tag{4.1}$$

$$\int\limits_{\mathbb{R}^3} \frac{\left|\sum_{\alpha\in II}\right|^2}{|x|^2} dx \leqslant C \sum_{\alpha\in II} |a_\alpha|^2, \tag{4.2}$$

and

$$\int\limits_{\mathbb{R}^3} \frac{\left|\sum_{\alpha\in III}\right|^2}{|x|^2} dx \leqslant C \sum_{\alpha\in III} |a_\alpha|^2. \tag{4.3}$$

In the following, we will only prove estimate 4.1, and the proofs of estimates 4.2 and 4.3 will be similar. To be more specific, we will use $\alpha_1$ and $x_1$ for estimate 4.1, $\alpha_2$ and $x_2$ for estimate 4.2, $\alpha_3$ and $x_3$ for estimate 4.3.

Define

$$u_{k,I}(\xi) = \sum_{\alpha\in I} a_\alpha X_{\alpha_1}(\xi_1) h_{\alpha_2}(\xi_2) h_{\alpha_3}(\xi_3)$$

then

$$\begin{aligned} &\|u_{k,I}\|_2^2 \\ =\ & \sum_{\alpha,\beta\in I} \int_{\mathbb{R}} d\xi_1 \int_{\mathbb{R}^2} d\xi_2 d\xi_3 a_\alpha \overline{a_\beta} X_{\alpha_1}(\xi_1) h_{\alpha_2}(\xi_2) h_{\alpha_3}(\xi_3) X_{\beta_1}(\xi_1) h_{\beta_2}(\xi_2) h_{\beta_3}(\xi_3) \\ =\ & \sum_{\alpha\in I} |a_\alpha|^2 \int_{\mathbb{R}} (X_{\alpha_1}(\xi_1))^2 d\xi_1 \\ \leqslant\ & 3 \sum_{\alpha\in I} |a_\alpha|^2, \end{aligned}$$

via formula 2.6.

Moreover,

$$\sum_{\alpha\in I} a_\alpha h_{\alpha_1}(\xi_1) h_{\alpha_2}(\xi_2) h_{\alpha_3}(\xi_3) = sign(\xi_1) \frac{\partial}{\partial \xi_1} u_{k,I}(\xi),\ \xi_1 \neq 0$$

yields

$$\begin{aligned} & (-i)^{-k}\sum_{\alpha\in I} a_\alpha h_{\alpha 1}(x_1)h_{\alpha_2}(x_2)h_{\alpha_3}(x_3) \\ = & H(t_1\check{u}_{k,I}(t))(x) \\ = & x_1 H(\check{u}_{k,I})(x) - \int_{-\infty}^{\infty} \check{u}_{k,even}(t,x_2,x_3)dt \end{aligned}$$

where $H$ is the Hilbert transform only with respect to the the first variable.

Hence we have

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{\left|\sum_{\alpha\in I}\right|^2}{|x|^2}dx &= \int_{\mathbb{R}^3} \frac{\left|x_1 H(\check{u}_{k,I})(x) - \int_{-\infty}^{\infty}\check{u}_{k,I}(t,x_2,x_3)dt\right|^2}{|x|^2}dx \\ &\leqslant 2\int_{\mathbb{R}^3} \frac{|x_1 H(\check{u}_{k,I})(x)|^2}{|x|^2}dx + 2\int_{\mathbb{R}^3} \frac{\left|\int_{-\infty}^{\infty}\check{u}_{k,I}(t,x_2,x_3)dt\right|^2}{|x|^2}dx_1dx_2dx_3 \\ &\leqslant 6\sum_{\alpha\in I}|a_\alpha|^2 + 2\pi\int_{\mathbb{R}^2}\left|\int_{-\infty}^{\infty}\check{u}_{k,I}(t,x_2,x_3)dt\right|^2 \frac{dx_2dx_3}{\sqrt{x_2^2+x_3^2}} \\ &= 6\sum_{\alpha\in I}|a_\alpha|^2 + 2\pi(MainTerm_I) \end{aligned}$$

where

$$\begin{aligned} MainTerm_I &= \int_{\mathbb{R}^2}\left|\int_{-\infty}^{\infty}\check{u}_{k,I}(t,x_2,x_3)dt\right|^2 \frac{dx_2dx_3}{\sqrt{x_2^2+x_3^2}} \\ &= \int_{\mathbb{R}^2}\left|\sum_{\alpha\in I}\int_{-\infty}^{\infty}dt\int_{\mathbb{R}^3} e^{it\xi_1}e^{ix_2\xi_2}e^{ix_3\xi_3}a_\alpha X_{\alpha_1}(\xi_1)h_{\alpha_2}(\xi_2)h_{\alpha_3}(\xi_3)d\xi\right|^2 \frac{dx_2dx_3}{\sqrt{x_2^2+x_3^2}} \\ &= \int_{\mathbb{R}^2}\left|\sum_{\alpha\in I}a_\alpha X_{\alpha_1}(0)(-i)^{\alpha_2+\alpha_3}h_{\alpha_2}(x_2)h_{\alpha_3}(x_3)\right|^2 \frac{dx_2dx_3}{\sqrt{x_2^2+x_3^2}} \\ &\leqslant C\int_{\mathbb{R}^2}\left||\nabla|^{\frac{1}{2}}\Big(\sum_{\alpha\in I}a_\alpha X_{\alpha_1}(0)(-i)^{\alpha_2+\alpha_3}h_{\alpha_2}(x_2)h_{\alpha_3}(x_3)\Big)\right|^2 dx_2dx_3 \quad \text{(Hardy's inequality)} \\ &\leqslant C\int_{\mathbb{R}^2}\left|(-\triangle+|(x_2,x_3)|^2)^{\frac{1}{4}}\Big(\sum_{\alpha\in I}a_\alpha X_{\alpha_1}(0)(-i)^{\alpha_2+\alpha_3}h_{\alpha_2}(x_2)h_{\alpha_3}(x_3)\Big)\right|^2 dx_2dx_3 \quad \text{(Lemma 1)} \\ &= C\sum_{\alpha\in I}|a_\alpha|^2 (X_{\alpha_1}(0))^2(2\alpha_2+2\alpha_3+2)^{\frac{1}{2}}. \end{aligned}$$

However, from Feldheim [11] and Busbridge [5], we know that given $\alpha_1$ even

$$\begin{aligned}(X_{\alpha_1}(0))^2 &= \frac{1}{4}(\int\limits_{-\infty}^{\infty} h_{\alpha_1}(t)dt)^2\\ &= \frac{\sqrt{2}}{4}\frac{2^{2\alpha_1}(\Gamma(\frac{1}{2}\alpha_1+\frac{1}{2}))^2}{2^{\alpha_1}\alpha_1!\sqrt{\pi}}\\ &\leqslant C\frac{1}{(\alpha_1)^{\frac{1}{2}}}\end{aligned}$$

by Stirling's formula. The above inequality shows

$$(X_{\alpha_1}(0))^2(\alpha_2+\alpha_3+1)^{\frac{1}{2}} \leqslant C\frac{(2\alpha_1+1)^{\frac{1}{2}}}{(\alpha_1)^{\frac{1}{2}}} \leqslant C$$

for $\alpha_1 \geqslant \frac{\alpha_2+\alpha_3}{2}$ and $\alpha_1 \neq 0$, or in other words, for $\alpha \in I$ and $k \neq 0$.

So

$$Mainterm_I \leqslant C\sum_{\alpha\in I}|a_\alpha|^2$$

i.e.

$$\int\limits_{\mathbb{R}^3}\frac{\left|\sum_{\alpha\in I}\right|^2}{|x|^2}dx \leqslant C\sum_{\alpha\in I}|a_\alpha|^2$$

**Remark 12.** *If we apply this procedure to the case when $n=2$ and $\delta=1, Mainterm_I$ will have $|x_2|^{-1}$ as a singularity which forces $MainTerm_I$ to be $\infty$ whenever there is some $a_a \neq 0$. But this procedure does also prove estimate 2.2 when $\delta < 1$ and hence theorem 3.*

## 5. An Application of Theorem 1 / Proof of Theorem 2

To obtain theorem 2, aside from theorem 1 and lemma 1, an interaction Morawetz inequality is needed.

### 5.1. Morawetz inequality.

As in [8], define

$$T_{00} = |u|^2$$

$$T_{0j} = T_{j0} = 2\,\mathrm{Im}\,\frac{\partial u}{\partial x_j}\overline{u}$$

and

$$T_{jk} = T_{kj} = 4\,\mathrm{Re}(u_k\overline{u_j}) - \delta_{jk}\triangle(|u|^2)$$

where $j$, $k$ mean summation from 1 to $n$. Then a direct computation shows that

$$\begin{aligned}\partial_t T_{00} + \partial_j T_{0j} &= 0,\\ \partial_t T_{k0} + \partial_j T_{kj} &= -2V_k|u|^2,\end{aligned}$$

for the equation

$$iu_t = -\triangle u + Vu.$$

Hence we have

$$\begin{aligned}
\partial_t M_0^a(t) &= 4\int_{\mathbb{R}^n} a_{kj}\operatorname{Re}(u_k\overline{u_j})dx - \int_{\mathbb{R}^n} \triangle a \triangle(|u|^2)dx - 2\int_{\mathbb{R}^n} a_k V_k |u|^2 dx. \\
&= 4\int_{\mathbb{R}^n} a_{kj}\operatorname{Re}(u_k\overline{u_j})dx - \int_{\mathbb{R}^n} \triangle a \triangle(|u|^2)dx + 2\int_{\mathbb{R}^n} a(x)V_{kk} |u|^2 dx \\
&\quad +2\int_{\mathbb{R}^n} a(x)(|u|^2)_k V_k dx
\end{aligned}$$

if we define

$$M_0^a(t) = \int_{\mathbb{R}^n} a_k(x) T_{0k}(t,x)dx$$

to be the Morawetz action corresponding to a suitable $a(x)$ which will be chosen momentarily.

Therefore, for equation 1.1 in the case $n = 9$, we in fact have

$$\begin{aligned}
&\int_0^{2\pi}\int_{\mathbb{R}^9} (\triangle^2 a)|u|^2 dxdt \\
= \; & 4\int_0^{2\pi}\int_{\mathbb{R}^9} a_{kj}(x)\operatorname{Re}(u_k\overline{u_j})dxdt \\
& +36\int_0^{2\pi}\int_{\mathbb{R}^9} a(x)|u|^2 dxdt \\
& +4\int_0^{2\pi}\int_{\mathbb{R}^9} a(x)V_k\operatorname{Re}(u\overline{u_k})dx
\end{aligned}$$

due to the facts that

$$\int_0^{2\pi} \partial_t M_0^a(t)dt = 0$$

and $V_{kk} = 18$.

The $a(x)$ we are going to pick is not non-strictly convex as in the usual cases in [8], but the follwing computation will help to simplify the technical problems arised

from that:

$$
\begin{aligned}
& 2\int_{\mathbb{R}^9} a_{kj}\operatorname{Re}(u_k\overline{u_j})dx \\
= & \int_{\mathbb{R}^9} a_{kj}(u_k\overline{u_j}+u_j\overline{u_k})dx \\
= & -\int_{\mathbb{R}^9} a_k(u_{kj}\overline{u_j}+u_k\overline{u_{jj}}+u_{jj}\overline{u_k}+u_j\overline{u_{kj}})dx \\
= & -\int_{\mathbb{R}^9} a_k(u_{kj}\overline{u_j}+u_j\overline{u_{kj}})dx-\int_{\mathbb{R}^9} a_k(u_k\overline{\triangle u}+\overline{u_k}\triangle u)dx \\
= & -\int_{\mathbb{R}^9} a_k(|u_j|^2)_k dx+\int_{\mathbb{R}^9} a(u_{kk}\overline{\triangle u}+u_k\overline{\triangle u_k}+\overline{u_{kk}}\triangle u+\overline{u_k}\triangle u_k)dx \\
= & \int_{\mathbb{R}^9} a_{kk}\left|u_j\right|^2 dx+2\int_{\mathbb{R}^9} a\left|\triangle u\right|^2 dx+\int_{\mathbb{R}^9} a(u_k\overline{\triangle u_k}+\overline{u_k}\triangle u_k)dx \\
= & \int_{\mathbb{R}^9} \triangle a\left|\nabla u\right|^2 dx+2\int_{\mathbb{R}^9} a\left|\triangle u\right|^2 dx+\int_{\mathbb{R}^9} a(\triangle\left|u_k\right|^2-2\left|\nabla u_k\right|^2)dx \\
= & 2\int_{\mathbb{R}^9} \triangle a\left|\nabla u\right|^2 dx+2\int_{\mathbb{R}^9} a\left|\triangle u\right|^2 dx-2\int_{\mathbb{R}^9} a\left|\nabla^2 u\right|^2 dx.
\end{aligned}
$$

So

$$
\begin{aligned}
\int_0^{2\pi}\int_{\mathbb{R}^9}(\triangle^2 a)\left|u\right|^2 dxdt = \; & 4\int_0^{2\pi}\int_{\mathbb{R}^9}\triangle a\left|\nabla u\right|^2 dxdt \\
& +4\int_0^{2\pi}\int_{\mathbb{R}^9} a(x)\left|\triangle u\right|^2 dxdt \\
& -4\int_0^{2\pi}\int_{\mathbb{R}^9} a(x)\left|\nabla^2 u\right|^2 dxdt \\
& +36\int_0^{2\pi}\int_{\mathbb{R}^9} a(x)\left|u\right|^2 dxdt \\
& +4\int_0^{2\pi}\int_{\mathbb{R}^9} a(x)V_k\operatorname{Re}(u\overline{u_k})dxdt
\end{aligned}
\tag{5.1}
$$

If we select

$$
a(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)=C\frac{1}{\left|\mathbf{x}_1-\mathbf{x}_2\right|^2+\left|\mathbf{x}_1-\mathbf{x}_3\right|^2+\left|\mathbf{x}_2-\mathbf{x}_3\right|^2}
\tag{5.2}
$$

where $C$ is a suitable positive constant, then

$$
\triangle^2 a=\delta(\mathbf{x}_1-\mathbf{x}_2)\delta(\mathbf{x}_2-\mathbf{x}_3),
$$

$$
\triangle a(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)=-C\frac{1}{(\left|\mathbf{x}_1-\mathbf{x}_2\right|^2+\left|\mathbf{x}_1-\mathbf{x}_3\right|^2+\left|\mathbf{x}_2-\mathbf{x}_3\right|^2)^2}<0,
$$

and relation 5.1 reads

$$
\begin{aligned}
\int_0^{2\pi}\int_{\mathbb{R}^3} |u(\mathbf{x},\mathbf{x},\mathbf{x},t)|^2 \, d\mathbf{x}dt &\leqslant 4\int_0^{2\pi}\int_{\mathbb{R}^9} a\left|\triangle u\right|^2 dxdt + 36\int_0^{2\pi}\int_{\mathbb{R}^9} a(x)\left|u\right|^2 dxdt \\
&\quad +4\int_0^{2\pi}\int_{\mathbb{R}^9} a(x)V_k \operatorname{Re}(u\overline{u_k})dxdt \\
&= 4A+36B+4D
\end{aligned}
$$

**Remark 13.** *Formula 5.2 is from Klainerman and Machedon's private communication*[16]. *Thanks to Machedon for sharing this computation.*

To prove estimate 1.7, it will suffice to show that $A, B$,and $D$ are majorized by $\left\|(-\triangle+|x|^2)f\right\|_2^2$.

### 5.2. Estimates for $A, B$, and $D$.

$$
\begin{aligned}
A &= \int_0^{2\pi}\int_{\mathbb{R}^9} a\left|\triangle u\right|^2 dxdt \\
&= C\int_0^{2\pi}\int_{\mathbb{R}^9} \frac{\left|\triangle u\right|^2}{\left|\mathbf{x}_1-\mathbf{x}_2\right|^2+\left|\mathbf{x}_1-\mathbf{x}_3\right|^2+\left|\mathbf{x}_2-\mathbf{x}_3\right|^2}dxdt \\
&\leqslant C\int_0^{2\pi}\int_{\mathbb{R}^9} \frac{\left|\triangle u\right|^2}{\left|\mathbf{x}_1-\mathbf{x}_2\right|^2}dxdt.
\end{aligned}
$$

due to the well-known change of variables $\mathbf{x}_1 \to \frac{\mathbf{x}_1-\mathbf{x}_2}{\sqrt{2}}, \mathbf{x}_2 \to \frac{\mathbf{x}_1+\mathbf{x}_2}{\sqrt{2}}$ which is compatible with $(-\triangle)$ and $(-\triangle+|x|^2)$, we only need to estimate

$$
\begin{aligned}
&\int_0^{2\pi}\int_{\mathbb{R}^9} \frac{\left|\triangle u\right|^2}{\left|\mathbf{x}_1\right|^2}dxdt \qquad (5.3)\\
&\leqslant C\int_0^{2\pi}\int_{\mathbb{R}^9} \frac{\left|(-\triangle+|x|^2)u\right|^2}{\left|\mathbf{x}_1\right|^2}dxdt + C\int_0^{2\pi}\int_{\mathbb{R}^9} \frac{\left|\left|x\right|^2 u\right|^2}{\left|\mathbf{x}_1\right|^2}dxdt \\
&\leqslant C\left\|(-\triangle+|x|^2)f\right\|_2^2 + C\int_0^{2\pi}\int_{\mathbb{R}^9} \frac{\left|\mathbf{x}_1\right|^4\left|u\right|^2}{\left|\mathbf{x}_1\right|^2}dxdt + C\int_0^{2\pi}\int_{\mathbb{R}^9} \frac{\left|\left|(\mathbf{x}_2,\mathbf{x}_3)\right|^2 u\right|^2}{\left|\mathbf{x}_1\right|^2}dxdt \\
&\leqslant C\left\|(-\triangle+|x|^2)f\right\|_2^2 + C\left\|\nabla\hat{f}\right\|_2^2 + E \\
&\leqslant C\left\|(-\triangle+|x|^2)f\right\|_2^2 + E
\end{aligned}
$$

where

$$
\begin{aligned}
E &\leqslant C\int\limits_0^{2\pi}\int_{\mathbb{R}^9}\frac{\left|(-\triangle_{\mathbf{x}_2,\mathbf{x}_3}+|(\mathbf{x}_2,\mathbf{x}_3)|^2)u\right|^2}{|\mathbf{x}_1|^2}dxdt+C\int\limits_0^{2\pi}\int_{\mathbb{R}^9}\frac{|\triangle_{\mathbf{x}_2,\mathbf{x}_3}u|^2}{|\mathbf{x}_1|^2}dxdt\\
&\leqslant C\int\limits_0^{2\pi}\int_{\mathbb{R}^9}\frac{\left|(-\triangle_{\mathbf{x}_2,\mathbf{x}_3}+|(\mathbf{x}_2,\mathbf{x}_3)|^2)u\right|^2}{|\mathbf{x}_1|^2}dxdt\\
&\leqslant C\left\|(-\triangle+|x|^2)f\right\|_2^2
\end{aligned}
$$

due to lemma 1 and theorem 1.

Then it is easy to see that

$$
\begin{aligned}
B &= 36\int\limits_0^{2\pi}\int_{\mathbb{R}^9}a(x)\,|u|^2\,dxdt\\
&\leqslant C\int\limits_0^{2\pi}\int_{\mathbb{R}^9}\frac{|u|^2}{|\mathbf{x}_1-\mathbf{x}_2|^2}dxdt\\
&\leqslant C\,\|f\|_2^2
\end{aligned}
$$

because of theorem 1 and change of variables.

The only term left over is

$$
D=\int\limits_0^{2\pi}\int_{\mathbb{R}^9}a(x)V_k\operatorname{Re}(u\overline{u_k})dxdt.
$$

A typical term in the sum reads

$$
\begin{aligned}
&\left|\int\limits_0^{2\pi}\int_{\mathbb{R}^9}a(x)V_1\operatorname{Re}(u\overline{u_1})dx\right|\\
\leqslant\;&\left(4\int\limits_0^{2\pi}\int_{\mathbb{R}^9}a(x)x_1^2\,|u|^2\,dxdt\right)^{\frac12}\left(\int\limits_0^{2\pi}\int_{\mathbb{R}^9}a(x)\left|\frac{\partial}{\partial x_1}u\right|^2dxdt\right)^{\frac12}\\
\leqslant\;&C\left\|(-\triangle+|x|^2)^{\frac12}f\right\|_2^2
\end{aligned}
$$

using the same method as in estimate 5.3.

Hence we conclude

$$
\int\limits_0^{2\pi}\int\limits_{\mathbb{R}^3}|u(\mathbf{x},\mathbf{x},\mathbf{x},t)|^2\,d\mathbf{x}dt\leqslant C\left\|(-\triangle+|x|^2)f\right\|_2^2
$$

which is theorem 2.

**Remark 14.** *If we choose not to ignore* $\int\limits_0^{2\pi}\int_{\mathbb{R}^9}\triangle a\left|\nabla u\right|^2dxdt$ *and* $-\int\limits_0^{2\pi}\int_{\mathbb{R}^9}a(x)\left|\nabla^2 u\right|^2dxdt$ *in relation 5.1, then in fact we have proven two additional Kato type smoothing estimates:*

$$\int\limits_0^{2\pi}\int_{\mathbb{R}^9}\frac{\left|\nabla u\right|^2}{(\left|\mathbf{x}_1-\mathbf{x}_2\right|^2+\left|\mathbf{x}_1-\mathbf{x}_3\right|^2+\left|\mathbf{x}_2-\mathbf{x}_3\right|^2)^2}dxdt\leqslant C\left\|(-\triangle+|x|^2)f\right\|_2^2$$

*and*

$$\int\limits_0^{2\pi}\int_{\mathbb{R}^9}\frac{\left|\nabla^2 u\right|^2}{\left|\mathbf{x}_1-\mathbf{x}_2\right|^2+\left|\mathbf{x}_1-\mathbf{x}_3\right|^2+\left|\mathbf{x}_2-\mathbf{x}_3\right|^2}dxdt\leqslant C\left\|(-\triangle+|x|^2)f\right\|_2^2.$$

## 6. Appendix: Proof of Lemma 5 / Computation of the $L^2$ norms of the "antiderivatives" of Hermite functions

In this section, we prove lemma 5 which yields the precise controlling constants. But we shall first prove that there exits a $C>0$ s.t.

$$\|X_k\|_2^2\leqslant C,\ \forall k$$

before we delve into the proof of lemma 5 which consists of many special function techniques.

### 6.1. Proof of the $L^2$ boundedness.

**Lemma 6.** [24] *We have the following creation and annihilation relations*

$$(-\frac{d}{dx}+x)\tilde{h}_k(x)=\tilde{h}_{k+1}(x)$$

$$(\frac{d}{dx}+x)\tilde{h}_k(x)=2k\tilde{h}_{k-1}(x)$$

*where* $\tilde{h}_k(x)=\frac{1}{c_k}h_k(x)$, *and* $c_k=(\frac{1}{2^kk!\sqrt{\pi}})^{\frac{1}{2}}$ *is the normalization constant, i.e.* $\tilde{h}_k(x)$ *is the unnormalized Hermite function of degree* $k$. *In this spirit, one has:*

$$\tilde{h}_{k+1}(x)=-2\frac{d}{dx}\tilde{h}_k(x)+2k\tilde{h}_{k-1}(x) \tag{6.1}$$

*or with the normalization factors*

$$h_{k+1}(x)=-\sqrt{\frac{2}{k+1}}\frac{d}{dx}h_k(x)+\sqrt{\frac{k}{k+1}}h_{k-1}(x). \tag{6.2}$$

We will only consider the even case

$$V_{2k}=\frac{\|X_{2k}\|_2^2}{2}=\int\limits_0^{\infty}(\int_x^{\infty}h_{2k}(t)dt)^2dx,$$

since the odd case is similar. Iterating relation 6.2 yields

$$h_{2k}(t)=\sum_{i=0}^{k-1}b_i\frac{d}{dt}h_{2k-1-2i}(t)+dh_0(t) \tag{6.3}$$

because

$$
\begin{aligned}
h_{2k}(x) &= -\sqrt{\frac{2}{2k}}\frac{d}{dx}h_{2k-1}(x) + \sqrt{\frac{2k-1}{2k}}h_{2k-2}(x) \\
h_{2k-2}(x) &= -\sqrt{\frac{2}{2k-2}}\frac{d}{dx}h_{2k-3}(x) + \sqrt{\frac{2k-3}{2k-2}}h_{2k-4}(x) \\
&\quad \ldots \\
h_4(x) &= -\sqrt{\frac{2}{4}}\frac{d}{dx}h_3(x) + \sqrt{\frac{3}{4}}h_2(x) \\
h_2(x) &= -\sqrt{\frac{2}{2}}\frac{d}{dx}h_1(x) + \sqrt{\frac{1}{2}}h_0(x).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int_0^\infty \left(\int_x^\infty h_{2k}(t)dt\right)^2 dx &\leqslant 2\int_0^\infty \left(\sum_{i=0}^{k-1} b_i h_{2k-1-2i}(x)\right)^2 dx + 2d^2\int_0^\infty \left(\int_x^\infty h_0(t)dt\right)^2 dx \\
&\leqslant 2\int_{-\infty}^\infty \left(\sum_{i=0}^{k-1} b_i h_{2k-1-2i}(x)\right)^2 dx + 2d^2\int_0^\infty \left(\int_x^\infty h_0(t)dt\right)^2 dx \\
&= 2(\sum_{i=0}^{k-1}|b_i|^2 + d^2\int_0^\infty \left(\int_x^\infty h_0(t)dt\right)^2 dx),
\end{aligned}
$$

where

$$
\begin{aligned}
\sum_{i=0}^{k-1}|b_i|^2 &= \frac{2}{2k} + \frac{2k-1}{2k}\frac{2}{2k-2} + \ \ldots\ + \frac{2k-1}{2k}\cdots\frac{2}{2} \\
&= \frac{1}{k} + \frac{1}{k}\frac{2k-1}{2k-2} + \ \ldots\ + \frac{1}{k}\frac{2k-1}{2k-2}\cdots\frac{3}{2} \\
&= \frac{1}{k}\sum_{i=0}^{k-1}\left(\prod_{l=0}^{i}(1+\frac{1}{2k-2l})\right).
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&\ln\prod_{l=0}^{i}(1+\frac{1}{2k-2l}) \\
\sim\ &\sum_{l=0}^{i}\frac{1}{2k-2l} \\
\sim\ &\frac{1}{2}\ln\frac{k-1}{k-i}
\end{aligned}
$$

which implies

$$\begin{aligned}
&\sum_{i=0}^{k-1}|b_i|^2\\
\sim\ & \frac{1}{k}\sum_{i=0}^{k-1}(\frac{k-1}{k-i})^{\frac{1}{2}}\\
\leqslant\ & \frac{1}{k}\sum_{i=0}^{k-1}(\frac{k}{k-i})^{\frac{1}{2}}\\
\leqslant\ & C\frac{1}{k^{\frac{1}{2}}}\int_0^k(\frac{1}{k-x})^{\frac{1}{2}}dx\\
\leqslant\ & C
\end{aligned}$$

i.e

$$\|X_{2k}\|_2^2 \leqslant C.$$

**Remark 15.** *For the odd case, formula 6.3 will read*

$$h_{2k+1}(t)=\sum_{i=0}^{k-1}b_i\frac{d}{dt}h_{2k-2i}(t)+dh_1(t).$$

6.2. **Proof of equalities 2.5 and 2.6.** Below we will refer to the following lemmas as well as lemma 6.

**Lemma 7.** *Write the degree $k$ Hermite polynomial $e^{\frac{x^2}{2}}\tilde{h}_k(x)$ as $H_k$,*

$$H_k(x)=\sum_{i=0}^{[\frac{k}{2}]}\frac{k!}{i!(k-2i)!}(-1)^i(2x)^{k-2i}$$

*then every polynomial $p(x)$ of degree $\leqslant i$ is a finite linear combination of $H_k, k\leqslant i$,*

$$p(x)=\sum_{k=0}^{i}(\int_{\mathbb{R}}\tilde{h}_k(x)p(x)e^{-\frac{1}{2}x^2}dx)H_k(x)$$

*In particular, given any polynomial $p(x)$ of degree$<k$, we have:*

$$\int_{\mathbb{R}}\tilde{h}_k(x)p(x)e^{-\frac{1}{2}x^2}dx=0$$

*Proof.* The first part of the statment is a well-known fact. To prove the second part, one only needs to notice that $p(x)e^{-\frac{1}{2}x^2}$ is a finite linear combination of $\tilde{h}_i(x)=H_ie^{-\frac{1}{2}x^2}, i<k$, and then apply orthogonality. $\square$

**Lemma 8.** [24]*If we define the degree $k$ Lagurre polynomial of type $\alpha$ by*

$$e^{-x}x^{\alpha}L_k^{\alpha}(x)=\frac{1}{k!}\frac{d^k}{dx^k}(e^{-x}x^{\alpha}),$$

*then*

$$H_{2k+1}=(-1)^k2^{2k+1}k!L_k^{\frac{1}{2}}(x^2)x. \tag{6.4}$$

**Remark 16.** *Formula 6.4 is (1.1.53) in Thangavelu* [24]*. He missed a factor* 2 *on the right hand side. One can refer to page 1001 of* [12].

At this point we can give the proof of formula 2.5

6.2.1. *Proof of the odd formula 2.5.* By relation 6.1, we have

$$\tilde{h}_{2k+1}(x) = -2\frac{d}{dx}\tilde{h}_{2k}(x) + 4k\tilde{h}_{2k-1}(x)$$

and hence

$$\left(\int_{-\infty}^{x}\tilde{h}_{2k+1}(t)dt\right)^2 = 4(\tilde{h}_{2k}(x))^2 - 16k\tilde{h}_{2k}(x)\cdot\int_{-\infty}^{x}\tilde{h}_{2k-1}(t)dt + 16k^2\left(\int_{-\infty}^{x}\tilde{h}_{2k-1}(t)dt\right)^2$$

or with the normalization factors

$$(X_{2k+1}(x))^2 = \frac{2}{2k+1}(h_{2k}(x))^2 - Junk(x) + \frac{2k}{(2k+1)}(X_{2k-1}(x))^2$$

where

$$Junk(x) = \frac{16k}{(c_{2k+1})^2}\tilde{h}_{2k}(x)\cdot\int_{-\infty}^{x}\tilde{h}_{2k-1}(t)dt.$$

So

$$I_{2k+1} = \frac{2}{2k+1} + \int_{\mathbb{R}} Junk(x)dx + \frac{2k}{2k+1}I_{2k-1}.$$

where

$$I_{2k+1} = \|X_{2k+1}\|_2^2$$

which is our target.

But $\tilde{h}_{2k-1}(t) = e^{-\frac{1}{2}x^2}(\sum_1^k b_{2i-1}x^{2i-1})$, so $\int_{-\infty}^{x}\tilde{h}_{2k-1}(t)dt = e^{-\frac{1}{2}x^2}(\sum_{i=0}^{k-1} l_{2i}x^{2i})$, which implies $\int_{\mathbb{R}} Junk(x)dx = 0$ by lemma 7. Hence

$$I_{2k+1} = \frac{2}{2k+1} + \frac{2k}{2k+1}I_{2k-1}. \tag{6.5}$$

The equalities that

$$I_1 = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{x} h_1(t)dt\right)^2 dx = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x} 2xe^{-\frac{1}{2}x^2}dt\right)^2 dx = 2$$

and relation 6.5 tell us

$$I_{2k+1} = 2.$$

6.2.2. *Proof of the even formula 2.6.* Applying relation 6.2 again, we have

$$\begin{aligned}
&V_{2k+2}\\
=\;& \frac{1}{2}\frac{1}{k+1} + \frac{\sqrt{2}\sqrt{2k+1}}{k+1}\int_0^{\infty} h_{2k+1}(x)\left(\int_x^{\infty} h_{2k}(t)dt\right)dx + \frac{2k+1}{2k+2}V_{2k}\\
=\;& \frac{1}{2}\frac{1}{k+1} - \frac{\sqrt{2}\sqrt{2k+1}}{k+1}\int_0^{\infty}\left(\frac{d}{dx}\int_x^{\infty} h_{2k+1}(t)dt\right)\left(\int_x^{\infty} h_{2k}(t)dt\right)dx + \frac{2k+1}{2k+2}V_{2k}
\end{aligned}$$

Just as the odd case, we are concerned with the middle term and would like to have an explicit formula for it. Integrating by parts once, we have

$$\begin{aligned}
&\int\limits_0^\infty \left(\frac{d}{dx}\int_x^\infty h_{2k+1}(t)dt\right)\left(\int_x^\infty h_{2k}(t)dt\right)dx \\
= \quad & -\left(\int_0^\infty h_{2k+1}(t)dt\right)\left(\int_0^\infty h_{2k}(t)dt\right)+\int\limits_0^\infty \left(\int_x^\infty h_{2k+1}(t)dt\right)h_{2k}(x)dx.
\end{aligned}$$

Recall that we already know

$$\begin{aligned}
\int_0^\infty h_{2k}(t)dt &= \frac{1}{2}\frac{2^{\frac{1}{2}}2^{2k}\Gamma(\frac{1}{2}(2k)+\frac{1}{2})}{\sqrt{2^{2k}(2k)!\sqrt{\pi}}} \\
&= \frac{2^k\Gamma(k+\frac{1}{2})}{\sqrt{(2k)!\sqrt{\pi}}} \\
&= \frac{2^{-k+\frac{1}{2}}(\sqrt{\pi})^{\frac{1}{2}}\Gamma(2k)}{\Gamma(k)\sqrt{(2k)!}}
\end{aligned}$$

from Feldheim [11], Busbridge [5] and the well-known formula for the gamma function

$$\Gamma(z+\frac{1}{2})=\frac{2^{1-2z}\sqrt{\pi}\Gamma(2z)}{\Gamma(z)},$$

so we would like to compute $\int_0^\infty h_{2k+1}(t)dt$ and $\int\limits_0^\infty (\int_x^\infty h_{2k+1}(t))h_{2k}(x)dx$. Using lemma 8, we have

$$\begin{aligned}
\int_0^\infty h_{2k+1}(t)dt &= \frac{1}{c_{2k+1}}(-1)^k 2^{2k+1}k!\int_0^\infty L_k^{\frac{1}{2}}(x^2)e^{-\frac{x^2}{2}}xdx \\
&= \frac{(-1)^k 2^{2k+1}k!}{2c_{2k+1}}\int_0^\infty L_k^{\frac{1}{2}}(u)e^{-\frac{u}{2}}du \\
&= \frac{(-1)^k 2^{2k+1}k!}{c_{2k+1}}\sum_{i=0}^k \binom{\frac{1}{2}+i-1}{i}(-1)^{k-i} \\
&= \frac{2^{k+1}k!}{\sqrt{2(2k+1)!\sqrt{\pi}}}\sum_{i=0}^k \binom{\frac{1}{2}+i-1}{i}(-1)^i
\end{aligned}$$

where the integral part, which has been worked out in page 809 of [12], is that

$$\int_0^\infty L_k^\alpha(u)e^{-\beta u}du=\sum_{i=0}^k\binom{\alpha+i-1}{i}\frac{(\beta-1)^{k-i}}{\beta^{k-i+1}}.$$

Hence

$$
\begin{aligned}
&\left(\int_0^\infty h_{2k+1}(t)dt\right)\left(\int_0^\infty h_{2k}(t)dt\right)\\
= &\quad \frac{2^{-k+\frac{1}{2}}(\sqrt{\pi})^{\frac{1}{2}}\Gamma(2k)}{\Gamma(k)\sqrt{(2k)!}}\frac{2^{k+1}k!}{\sqrt{2(2k+1)!\sqrt{\pi}}}\sum_{i=0}^{k}\binom{\frac{1}{2}+i-1}{i}(-1)^i\\
= &\quad \frac{1}{\sqrt{2k+1}}\sum_{i=0}^{k}\binom{\frac{1}{2}+i-1}{i}(-1)^i\\
= &\quad \frac{1}{\sqrt{2k+1}}\sum_{i=0}^{k}\binom{-\frac{1}{2}}{i},
\end{aligned}
$$

due to the identity

$$\binom{\alpha}{k}=\binom{k-\alpha-1}{k}(-1)^k.$$

For the last term, we have

$$
\begin{aligned}
\int\limits_0^\infty\left(\int_x^\infty h_{2k+1}(t)\right)h_{2k}(x)dx &= \frac{1}{2}\int\limits_{-\infty}^{\infty}\left(\int_x^\infty h_{2k+1}(t)dt\right)h_{2k}(x)dx\\
&= \frac{1}{2}\frac{(\text{the coefficient of } x^{2k}e^{-\frac{x^2}{2}} \text{ in } \int_x^\infty h_{2k+1}(t)dt)}{(\text{the coefficient of } x^{2k}e^{-\frac{x^2}{2}} \text{ in } h_{2k}(x))}\\
&= \frac{1}{2}\frac{\frac{2^{2k+1}}{c_{2k+1}}}{\frac{2^{2k}}{c_{2k}}}\\
&= \frac{1}{\sqrt{2}\sqrt{2k+1}},
\end{aligned}
$$

via lemma 7.

At long last we have

$$V_{2k+2}=-\frac{1}{2k+2}+\frac{\sqrt{2}}{k+1}\sum_{i=0}^{k}\binom{-\frac{1}{2}}{i}+\frac{2k+1}{2k+2}V_{2k}.$$

Since

$$\frac{1}{k+1}\sum_{i=0}^{k}\binom{-\frac{1}{2}}{i}+\frac{2k+1}{2k+2}\sum_{i=0}^{k}\binom{\frac{1}{2}}{i}=\sum_{i=0}^{k+1}\binom{\frac{1}{2}}{i},$$

a straight forward induction gives us formula 2.6. This concludes the proof of lemma 5.

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